

# New bounds for expected maxima of fractional Brownian motion

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## Abstract

For the fractional Brownian motion  $B^H$  with the Hurst parameter value  $H$  in  $(0, 1/2)$ , we derive new upper and lower bounds for the difference between the expectations of the maximum of  $B^H$  over  $[0, 1]$  and the maximum of  $B^H$  over the discrete set of values  $in^{-1}$ ,  $i = 1, \dots, n$ . We use these results to improve our earlier upper bounds for the expectation of the maximum of  $B^H$  over  $[0, 1]$  and derive new upper bounds for Pickands' constant.

*Key words and phrases:* fractional Brownian motion, convergence rate, discrete time approximation, Pickands' constant.

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## 1 Introduction and main results

Let  $B^H = (B_t^H)_{t \geq 0}$  be a fractional Brownian motion (fBm) process with Hurst parameter  $H \in (0, 1)$ , i.e. a zero-mean continuous Gaussian process with the covariance function  $\mathbf{E} B_s^H B_t^H = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H})$ ,  $s, t \geq 0$ . Equivalently, the last condition can be stated as  $B_0^H = 0$  and

$$\mathbf{E}(B_s^H - B_t^H)^2 = |s - t|^{2H}, \quad s, t \geq 0. \quad (1)$$

Recall that the Hurst parameter  $H$  characterises the type of the dependence of the increments of the fBm. For  $H \in (0, \frac{1}{2})$  and  $H \in (\frac{1}{2}, 1)$ , the increments of  $B^H$

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are respectively negatively and positively correlated, whereas the process  $B^{1/2}$  is the standard Brownian motion which has independent increments. The fBm processes are important construction blocks in various application areas, the ones with  $H > \frac{1}{2}$  being of interest as their increments exhibit long-range dependence, while it was shown recently that fBm's with  $H < \frac{1}{2}$  can be well fitted to real life telecommunications and financial data (see, e.g., [2, 3]). For detailed exposition of the theory of fBm processes, we refer the reader to [4, 9, 8] and references therein.

Computing the value of the expected maximum

$$M^H := \mathbf{E} \max_{0 \leq t \leq 1} B_t^H$$

is an important question arising in a number of applied problems, such as finding the likely magnitude of the strongest earthquake to occur this century in a given region or the speed of the strongest wind gust a tall building has to withstand during its lifetime etc. For the standard Brownian motion  $B^{1/2}$ , the exact value of the expected maximum is  $\sqrt{\pi/2}$ , whereas for all other  $H \in (0, 1)$  no closed-form expressions for the expectation are known. In the absence of such results, one standard approach to computing  $M^H$  is to evaluate instead its approximation

$$M_n^H := \mathbf{E} \max_{1 \leq i \leq n} B_{i/n}^H, \quad n \geq 1,$$

(which can, for instance, be done using simulations) together with the approximation error

$$\Delta_n^H := M^H - M_n^H.$$

Some bounds for  $\Delta_n^H$  were recently established in [5]. The main result of the present note is an improvement the following upper bound for  $\Delta_n^H$  obtained in Theorem 3.1 of that paper: for  $n \geq 2^{1/H}$ ,

$$\Delta_n^H \leq \frac{2(\ln n)^{1/2}}{n^H} \left( 1 + \frac{4}{n^H} + \frac{0.0074}{(\ln n)^{3/2}} \right).$$

We also obtain a new upper bound for the expected maximum  $M^H$  itself, which refines previously known results (see e.g. [5, 11]), and use it to derive an improved upper bound for the so-called Pickands' constant, which is the basic constant in the extreme value theory of Gaussian processes.

From now on, we always assume that  $H \in (0, \frac{1}{2})$ . The next theorem is the main result of the note. As usual,  $\lfloor x \rfloor$  and  $\lceil x \rceil$  denote the floor and the ceiling of the real number  $x$ .

**Theorem 1.** *For any  $n \geq 2$  and  $\alpha > 0$ , one has*

$$n^H \left( \frac{L}{(\ln n^H)^{1/2}} - 1 \right) \leq \frac{\Delta_n^H}{n^{-H}(\ln n)^{1/2}} \leq \frac{(1 - m^{-1})^H (1 + \alpha_n)^{1/2}}{1 - m^{-H} (1 + \alpha_n / (1 + \alpha_n))^{1/2}}, \quad (2)$$

where  $L = 1/\sqrt{4\pi e \ln 2} \approx 0.206 \dots$ ,  $m := 2 \vee \lfloor n^\alpha \rfloor$  and  $\alpha_n := \alpha \vee (\log_2 n)^{-1}$ .

Note that the left inequality in (2) actually holds for all  $H \in (0, 1)$ .

**Remark 1.** Choosing  $\alpha = \alpha(n) \rightarrow 0$  slowly enough as  $n \rightarrow \infty$  (one can take, e.g.,  $\alpha = (\ln \ln n)/\ln n$ ), we obtain from the upper bound in (2) that, for any fixed  $H \in (0, \frac{1}{2})$ , one has

$$\Delta_n^H \leq n^{-H}(\ln n)^{1/2}(1 + o(1)), \quad n \rightarrow \infty.$$

**Remark 2.** Recall that, in the case of the standard Brownian motion ( $H = \frac{1}{2}$ ), the exact asymptotics of  $\Delta_n^{1/2}$  are well-known and contain no logarithmic factor:

$$\Delta_n^{1/2} = n^{-1/2}(\beta + o(1)), \quad n \rightarrow \infty,$$

where  $\beta = -\zeta(1/2)/\sqrt{2\pi} = 0.5826\dots$  and  $\zeta(\cdot)$  is the Riemann zeta function (see [12]). The left inequality in (2) shows, however, that the factor  $(\ln n)^{1/2}$  in a “universal” upper bound of the form of the right inequality in (2) that would hold for all  $H \in (0, \frac{1}{2})$ , cannot be removed. Indeed, it follows from that inequality that, for any  $H \in (0, 1)$ , one has  $\Delta_n \geq (2/\ln 2 - 1)n^{-H}(\ln n)^{1/2}$  for  $n = \lfloor 2^{1/H} \rfloor$ . If, however, we restrict  $H$  to the interval  $(\varepsilon, \frac{1}{2})$  with a fixed  $\varepsilon \in (0, \frac{1}{2})$ , then it might be possible to improve the upper bound (perhaps to a bound of the form  $o((\ln n)^{1/2})n^{-H}$ ).

The next simple assertion enables one to use the upper bound obtained in Theorem 1 to get an upper bound for the approximation rate of the expectation of a function of the maximum of an fBm. Such a result is required, for instance, for bounding convergence rates when approximating Bayesian estimators in irregular statistical experiments (see, e.g., [10]).

Set

$$\overline{B}_1^H := \max_{0 \leq t \leq 1} B_t^H, \quad \overline{B}_{n,n}^H := \max_{1 \leq i \leq n} B_{i/n}^H, \quad \Delta_n^{H,f} := \mathbf{E}f(\overline{B}_1^H) - \mathbf{E}f(\overline{B}_{n,n}^H)$$

and, for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , denote its continuity modulus by

$$\omega_{\delta,h}(f) := \sup_{0 \leq s < t \leq (s+\delta) \wedge h} |f(s) - f(t)|, \quad h, \delta > 0.$$

**Corollary 1.** *Let  $f \geq 0$  be an arbitrary non-decreasing function on  $\mathbb{R}$  such that  $f(x) = o(\exp((x - M^H)^2/2))$  as  $x \rightarrow \infty$ . Then, for any number  $M > M^H$ ,*

$$\Delta_n^{H,f} \leq \omega_{\Delta_n^H, M}(f) + \int_M^\infty f(x)(x - M^H) \exp\{-(x - M^H)^2/2\} dx.$$

To roughly balance the contributions from the two terms in the bound, one may wish to choose  $M$  so that  $\exp\{-(M - M^H)^2/2\}$  would be of the same order of magnitude as  $\Delta_n^H$  (as for regular functions  $f$  that are mostly of interest in

applications are locally Lipschitz, so that  $\omega_{\delta,h}(f)$  admits a linear upper bound in  $\delta$ ). To that end, one can take  $M := M^H + (-2 \ln \Delta_n^H)^{1/2} + \text{const}$  (assuming that  $n$  is large enough so that  $\Delta_n^H < 1$ ). We will illustrate that in two special cases where  $f$  is the exponential function (this case corresponds to the above-mentioned applications from [10]) and a power function, respectively.

**Example 1.** Assume that  $f(x) = e^{ax}$  with a fixed  $a > 0$ , and that  $\Delta_n^H < 1$ . Choosing  $M := M^H + a + |2 \ln \Delta_n^H|^{1/2}$  we get

$$\omega_{\Delta_n^H, M}(f) \leq e^{aM} \Delta_n^H = \exp\{aM^H + a^2 + a|2 \ln \Delta_n^H|^{1/2}\} \Delta_n^H$$

and, setting  $y := x - M^H$  and using the well-known bound for the Mills' ratio for the normal distribution, obtain that

$$\begin{aligned} \int_M^\infty f(x)(x - M^H) \exp\{-(x - M^H)^2/2\} dx &= e^{aM^H} \int_{M-M^H}^\infty y e^{-y^2/2+ay} dy \\ &= e^{aM^H+a^2/2} \left[ \int_{M-M^H}^\infty (y-a) e^{-(y-a)^2/2} dy + a \int_{M-M^H}^\infty e^{-(y-a)^2/2} dy \right] \\ &\leq e^{aM^H+a^2/2} \left( 1 + \frac{a}{M - M^H - a} \right) e^{-(M-M^H-a)^2/2} \\ &= e^{aM^H+a^2/2} \left( 1 + \frac{a}{|2 \ln \Delta_n^H|^{1/2}} \right) \Delta_n^H. \end{aligned}$$

Therefore

$$\Delta_n^{H,f} \leq e^{aM^H+a^2/2} \left( 1 + e^{a^2/2+a|2 \ln \Delta_n^H|^{1/2}} + \frac{a}{|2 \ln \Delta_n^H|^{1/2}} \right) \Delta_n^H.$$

**Example 2.** For the function  $f(x) = x^p$ ,  $p \geq 1$ , one clearly has

$$\Delta_n^{H,f} \leq pM^{p-1} \Delta_n^H + \int_M^\infty x^p (x - M^H) \exp\{-(x - M^H)^2/2\} dx.$$

Observe that  $x^p = (x - M^H)^p \left( 1 + \frac{M^H}{x - M^H} \right)^p \leq (x - M^H)^p \left( \frac{M}{M - M^H} \right)^p$  for  $x \geq M$ , while, for any  $A > 0$ ,

$$\int_A^\infty z^{p+1} e^{-z^2/2} dz = - \int_A^\infty z^p d e^{-z^2/2} = A^p e^{-A^2/2} + p \int_A^\infty z^{p-1} e^{-z^2/2} dz,$$

where the last integral does not exceed  $A^{-2} \int_A^\infty z^{p+1} e^{-z^2/2} dz$ , so that

$$\int_A^\infty z^{p+1} e^{-z^2/2} dz \leq \frac{A^p e^{-A^2/2}}{1 - pA^{-2}} \quad \text{for } A^2 > p.$$

Hence, choosing  $A := M - M^H = |2 \ln \Delta_n^H|^{1/2}$ , we obtain that, for  $\Delta_n^H < e^{-p/2}$ ,

$$\Delta_n^{H,f} \leq (M^H + |2 \ln \Delta_n^H|^{1/2})^{p-1} \left( p + \frac{M^H + |2 \ln \Delta_n^H|^{1/2}}{1 - p|2 \ln \Delta_n^H|^{-1}} \right) \Delta_n^H.$$

Finally, in the next corollary we use Theorem 1 to improve the known upper bound  $M^H < 16.3H^{-1/2}$  for the expected maximum  $M^H$  from Theorem 2.1(ii) in [5].

**Corollary 2.** *Assume that  $H$  is such that  $2^{2/H}$  is integer. Then*

$$M^H < 1.695H^{-1/2}.$$

**Remark 3.** If  $2^{2/H}$  is not integer then, in the above formula, one can use instead of  $H$  the largest value  $\tilde{H} < H$  such that  $2^{2/\tilde{H}}$  is integer, i.e.  $\tilde{H} = 2/\log_2[2^{2/H}]$ . This is so since it follows from Sudakov–Fernique’s inequality (see e.g. Proposition 1.1 and Section 4 in [5]) that the expected maximum  $M^H$  is a non-increasing function of  $H$ .

**Remark 4.** Our new upper bound for  $M^H$  can be used to improve Shao’s upper bound from [11] for Pickands’ constant  $\mathcal{H}_a$ , which is a basic constant in the extreme value theory of Gaussian processes and is of interest in a number of applied problems. That constant appears in the asymptotic representation for the tail probability of the maxima of stationary Gaussian processes in the following way. Assume that  $(X_t)_{t \geq 0}$  is a stationary Gaussian process with zero mean and unit variance of which the covariance function  $r(v) := \mathbf{E}X_tX_{t+v}$ , satisfies the following relation: for some  $C > 0$  and  $a \in (0, 2]$ , one has  $r(t) = 1 - C|t|^a + o(|t|^a)$  as  $t \rightarrow 0$ . Then, for each fixed  $h > 0$  such that  $\sup_{\varepsilon \leq t \leq h} r(t) < 1$  for all  $\varepsilon > 0$ ,

$$\mathbf{P}\left(\sup_{0 \leq t \leq h} X_t > u\right) = C^{1/a} \mathcal{H}_a (2\pi)^{-1/2} e^{-u^2/2} u^{2/a-1} (h + o(1)), \quad u \rightarrow \infty.$$

It was shown in [11] that, for  $a \in (0, 1]$ ,

$$\mathcal{H}_a \leq (2^{-1/2} e a M^{a/2})^{2/a}.$$

Using our Corollary 2, we obtain the following new upper bound for Pickands’ constant:

$$\mathcal{H}_a < (21.23a)^{1/a}, \quad a \in (0, 1],$$

which is superior to Shao’s bound

$$\mathcal{H}_a \leq \left\{ a^{1/2} [0.77a^{1/2} + 2.41(8.8 - a \ln(0.4 + 2.5/a))^{1/2}] \right\}^{2/a}, \quad a \in (0, 1]$$

(see (1.5) in [11]). Thus, the ratio of our bound to Shao’s equals 0.344 when  $a = 0.9$  and is 0.046 when  $a = 0.3$ .

## 2 Proofs

*Proof of Theorem 1.* First we will prove the right inequality in (2). Let  $n_k := nm^k$ ,  $k \geq 0$ , where we set  $m := 2 \vee \lfloor n^\alpha \rfloor$ . It follows from the continuity of  $B^H$  and monotone convergence theorem that

$$\Delta_n^H = \sum_{k=0}^{\infty} (M_{n_{k+1}}^H - M_{n_k}^H). \quad (3)$$

Although this step is common with the proof of Theorem 3.1 in [5], the rest of the argument uses a different idea. Namely, we apply Chatterjee's inequality ([6]; see also Theorem 2.2.5 in [1]) which, in its general formulation, states the following. For any  $N$ -dimensional Gaussian random vectors  $X = (X_1, \dots, X_N)$ ,  $Y = (Y_1, \dots, Y_N)$  with common means:  $\mathbf{E}X_i = \mathbf{E}Y_i$  for  $1 \leq i \leq N$ , one has

$$\left| \mathbf{E} \max_{1 \leq i \leq N} X_i - \mathbf{E} \max_{1 \leq i \leq N} Y_i \right| \leq (\gamma \ln N)^{1/2}, \quad \gamma := \max_{1 \leq i < j \leq N} |d_{ij}(X) - d_{ij}(Y)|, \quad (4)$$

where, for a random vector  $Z \in \mathbb{R}^N$ , we set  $d_{ij}(Z) := \mathbf{E}(Z_i - Z_j)^2$ ,  $1 \leq i, j \leq N$ .

To be able to apply that inequality to the terms in the sum on the right-hand side of (3), for each  $k \geq 0$  we introduce auxiliary vectors  $X^k, Y^k \in \mathbb{R}^{n_{k+1}}$  by letting

$$X_i^k := B_{i/n_{k+1}}^H, \quad Y_i^k := B_{\lfloor i/m \rfloor / n_k}^H, \quad 1 \leq i \leq n_{k+1}.$$

Note that  $M_{n_{k+1}}^H = \max_{1 \leq i \leq n_{k+1}} X_i^k$  and  $M_{n_k}^H = \max_{1 \leq i \leq n_{k+1}} Y_i^k$ , so that now (4) is applicable. Next we will show that

$$\gamma^k := \max_{1 \leq i < j \leq n_{k+1}} |d_{ij}(X^k) - d_{ij}(Y^k)| \leq n_k^{-2H} (1 - m^{-1})^{2H}.$$

Indeed, one can clearly write down the representations  $i = a_i m + b_i$ ,  $j = a_j m + b_j$  with integer  $a_j \geq a_i \geq 0$  and  $1 \leq b_i, b_j \leq m$ , such that  $b_j > b_i$  when  $a_i = a_j$ . Then it follows from (1) that

$$d_{ij}(X^k) = \left( \frac{(a_j - a_i)m + b_j - b_i}{n_{k+1}} \right)^{2H}, \quad d_{ij}(Y^k) = \left( \frac{(a_j - a_i)m}{n_{k+1}} \right)^{2H}.$$

Since for  $2H \leq 1$  the function  $x \mapsto x^{2H}$ ,  $x \geq 0$ , is concave, it is also sub-additive, so that  $x^{2H} - y^{2H} \leq (x - y)^{2H}$  for any  $x \geq y \geq 0$ . Setting  $x := d_{ij}(X^k) \vee d_{ij}(Y^k)$  and  $y := d_{ij}(X^k) \wedge d_{ij}(Y^k)$ , this yields the desired bound

$$|d_{ij}(X^k) - d_{ij}(Y^k)| \leq \left( \frac{|b_i - b_j|}{n_{k+1}} \right)^{2H} \leq \left( \frac{m - 1}{n_{k+1}} \right)^{2H} = \frac{1}{n_k^{2H}} \left( 1 - \frac{1}{m} \right)^{2H}.$$

Now it follows from (4) that

$$\begin{aligned}
M_{n_{k+1}}^H - M_{n_k}^H &\equiv \mathbf{E} \max_{1 \leq i \leq n_{k+1}} X_i^k - \mathbf{E} \max_{1 \leq i \leq n_{k+1}} Y_i^k \\
&\leq (\gamma^k \ln n_{k+1})^{1/2} \leq \frac{(1 - m^{-1})^H}{n^H m^{kH}} (\ln n + (k+1) \ln m)^{1/2} \\
&\leq \frac{(\ln n)^{1/2}}{n^H} (1 - m^{-1})^H \frac{(1 + \alpha_n + \alpha_n k)^{1/2}}{m^{kH}},
\end{aligned}$$

recalling that  $\alpha_n = \alpha \vee (\log_2 n)^{-1}$ . The last bound together with (3) leads to

$$\Delta_n^H \leq \frac{(\ln n)^{1/2}}{n^H} (1 - m^{-1})^H \sum_{k=0}^{\infty} \frac{(1 + \alpha_n + \alpha_n k)^{1/2}}{m^{kH}}.$$

The sum of the series on the right hand side is exactly  $\alpha_n^{1/2} \Phi(m^{-H}, -\frac{1}{2}, 1 + \alpha_n^{-1})$ , where  $\Phi$  is the Lerch transcendent function. For our purposes, however, it will be convenient just to use the elementary bound  $(1 + \alpha_n + \alpha_n k)^{1/2} \leq (1 + \alpha_n)^{1/2} (1 + \alpha_n / (1 + \alpha_n))^{k/2}$ , to get

$$\Delta_n^H \leq \frac{(\ln n)^{1/2}}{n^H} \cdot \frac{(1 - m^{-1})^H (1 + \alpha_n)^{1/2}}{1 - m^{-H} (1 + \alpha_n / (1 + \alpha_n))^{1/2}}.$$

The right inequality in (2) is proved. To establish the left one, note that, on the one hand, it was shown in Theorem 2.1 [5] that  $M^H \geq LH^{-1/2}$  for all  $H \in (0, 1)$ .

On the other hand, it follows from Sudakov–Fernique’s inequality (see e.g. Proposition 1.1 in [5]) that, for any fixed  $n \geq 1$ , the quantity  $M_n^H$  is non-increasing in  $H$ , and it follows from Lemma 4.1 in [5] that

$$M_n^0 := \lim_{H \rightarrow 0} M_n^H = 2^{-1/2} \mathbf{E} \bar{\xi}_n, \quad \bar{\xi}_n := \max_{1 \leq i \leq n} \xi_i,$$

where  $\xi_i$  are i.i.d.  $N(0, 1)$ -distributed random variables. Furthermore, the last expectation admits the following upper bound:

$$\mathbf{E} \bar{\xi}_n \leq \sqrt{2 \ln n}, \quad n \geq 1. \quad (5)$$

Although that bound has been known for some time, we could not find a suitable literature reference or stable Internet link for it. So decided to include a short known proof thereof for completeness’ sake. By Jensen’s inequality, for any  $s \in \mathbb{R}$ ,

$$e^{s \mathbf{E} \bar{\xi}_n} \leq \mathbf{E} e^{s \bar{\xi}_n} = \mathbf{E} \max_{1 \leq i \leq n} e^{s \xi_i} \leq \mathbf{E} \sum_{1 \leq i \leq n} e^{s \xi_i} = \sum_{1 \leq i \leq n} \mathbf{E} e^{s \xi_i} = n e^{s^2/2},$$

so that  $\mathbf{E} \bar{\xi}_n \leq s^{-1} \ln n + s/2$ . Minimizing in  $s$  the expression on the right-hand side yields the desired bound (5).

From the above results, we obtain that

$$\begin{aligned} M^H - M_n^H &\geq M^H - M_n^0 \geq LH^{-1/2} - (\ln n)^{1/2} \\ &= n^{-H}(\ln n)^{1/2} \cdot n^H(L(H \ln n)^{-1/2} - 1), \end{aligned}$$

which completes the proof of Theorem 1.  $\square$

*Proof of Corollary 1.* Since  $f \geq 0$ , for any  $M > M^H$  we have

$$\begin{aligned} \Delta_n^{H,f} &\leq \mathbf{E}(f(\overline{B}_1^H) - f(\overline{B}_{n,n}^H); \overline{B}_1^H \leq M) + \mathbf{E}(f(\overline{B}_1^H); \overline{B}_1^H > M) \\ &\leq \omega_{\Delta_n^H, M}(f) + \int_M^\infty f(x) dF(x), \end{aligned} \quad (6)$$

where  $F(x) := \mathbf{P}(\overline{B}_1^H \leq x)$ . From the well-known Borell–TIS inequality for Gaussian processes (see, e.g., Theorem 2.1.1 in [1]) it follows that, for any  $u > 0$ ,

$$\mathbf{P}(\overline{B}_1^H - M^H > u) \leq e^{-u^2/2}.$$

Therefore, for any  $M > M^H$ , integrating by parts, using the assumed property that  $f(x) \exp\{-(x - M^H)^2/2\} \rightarrow 0$  as  $x \rightarrow \infty$ , and then again integrating by parts, we can write

$$\begin{aligned} \int_M^\infty f(x) dF(x) &= f(M)(1 - F(M)) + \int_M^\infty (1 - F(x)) df(x) \\ &\leq f(M) \exp\{-(M - M^H)^2/2\} + \int_M^\infty \exp\{-(x - M^H)^2/2\} df(x) \\ &= - \int_M^\infty f(x) d \exp\{-(x - M^H)^2/2\}. \end{aligned}$$

Together with (6) this establishes the assertion of Corollary 1.  $\square$

*Proof of Corollary 2.* Using Chatterjee's inequality (4) with the zero vector  $Y$ , we get for any  $n \geq 1$  the bound  $M_n^H \leq ((1 - n^{-2H}) \ln n)^{1/2}$ , so that we obtain from Theorem 1 that

$$\begin{aligned} M^H &\leq \Delta_n^H + ((1 - n^{-2H}) \ln n)^{1/2} \\ &< \left[ \frac{n^{-H}(1 + \alpha)^{1/2}}{1 - m^{-H}(1 + \alpha/(1 + \alpha))^{1/2}} + (1 - n^{-2H})^{1/2} \right] (\ln n)^{1/2}. \end{aligned}$$

Now choosing  $n := 4^{1/H}$  (which was assumed to be integer) and  $\alpha := 2$ , we get  $m = n^\alpha = 4^{2/H}$  and

$$M^H < H^{-1/2} \left[ \frac{4^{-1} 3^{1/2}}{1 - 16^{-1} (5/3)^{1/2}} + (1 - 16^{-1})^{1/2} \right] (\ln 4)^{1/2} < 1.695 H^{-1/2}.$$

$\square$

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